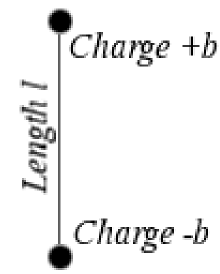
# *The Field of a Small Magnetic Dipole*

On this page I'll derive the field of a magnetic dipole.

The usual model of a magnetic dipole is a small current loop. If the loop has area $A$ and the current in the loop is $I$, then the magnitude of the dipole is $IA$, and its vector is perpendicular to the loop. However, this model is a little awkward to work with.

***Figure 1** -- Two-charge dipole:*

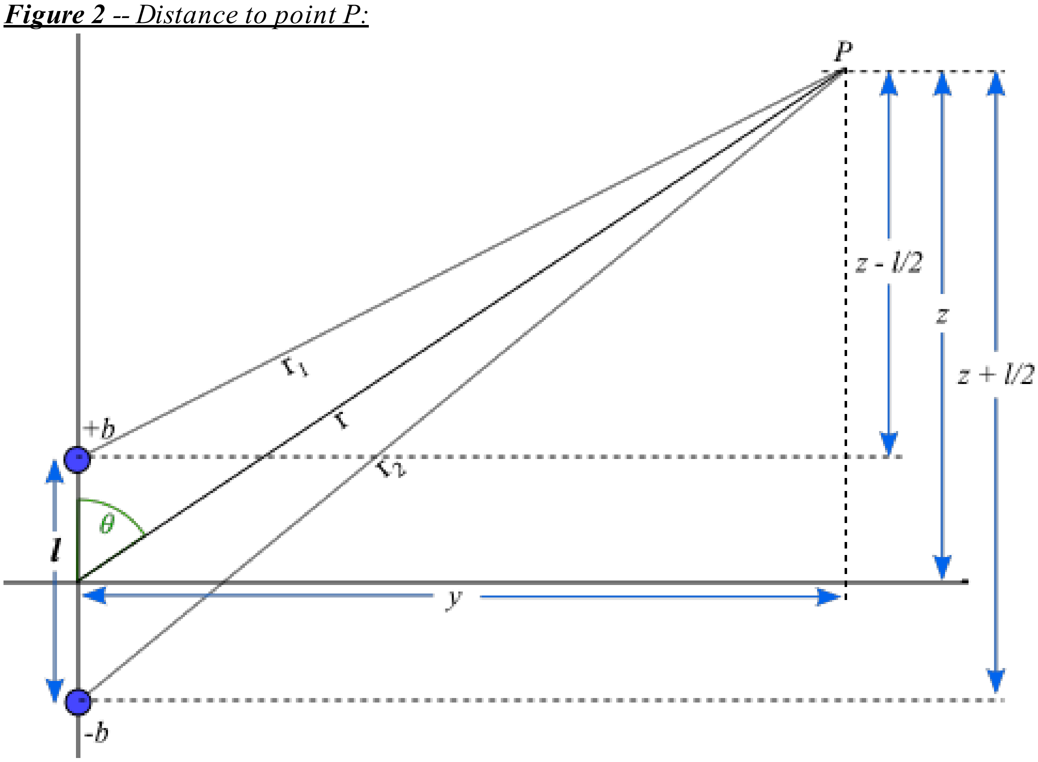



On our two-charge dipole page, we argue that a dipole model consisting of two charges, $+b$ and $-b$, separated by an infinitesimal distance $l$, produces the same field as an infinitesimal current loop dipole if $bl = IA = |\mu|$, where $\mu$ is the magnetic dipole moment, and is a vector. Since the two models produce the same field, we can use the two-charge dipole model to determine the field -- even though it's physically unrealizable, as there are no magnetic monopoles. (Of course, the dipole field will have the same form whether it's an electric or magnetic dipole -- and an electric dipole really does consist of two charges.)

Note that on this page, we're using $B$ and $\mu$ to mean the *vector* quantities representing the magnetic field and the dipole moment. We may also use an overarrow in some formulas to emphasize the fact that these are vectors.

For the remainder of this page, unless otherwise noted we'll assume that our "magnetic dipole" consists of two charges, as in figure 1. We'll further assume that the dipole is aligned with the $z$ axis. We wish to find the field at point $P$ as shown in figure 2. $P$ is assumed to lie in the $yz$ plane.

***Figure 2** -- Distance to point P:*



## *The Component $B_z$*

The field at point ***P*** in figure 2 will be the sum of the fields from charges +***b*** and -***b***. The distances between ***P*** and the origin and the two charges are:

$$
\begin{aligned}
r &= \sqrt{y^2 + z^2} \\
r_1 &= \sqrt{y^2 + \left(z - \tfrac{l}{2}\right)^2} \qquad (1) \\
r_2 &= \sqrt{y^2 + \left(z + \tfrac{l}{2}\right)^2}
\end{aligned}
$$

The field at ***P*** is just the sum of the fields due to each of the charges. For a point charge, if the radius vector at a particular location has angle θ with the ***z*** axis and the total field strength at that location is |***B***|, we have

$$(2) \quad B_{z[point]} = \left|B_{[point]}\right| \cos\theta$$

In this case, we're summing the fields of two point charges. Keeping in mind that the field around -***b*** points toward the charge rather than away from it, we have

$$
\begin{aligned}
B_z &= B_{z(+b)} + B_{z(-b)} \\
&= \frac{b}{r_1^2}\frac{z-\frac{l}{2}}{r_1} - \frac{b}{r_2^2}\frac{z+\frac{l}{2}}{r_2} \qquad (3) \\
&= b\left[\frac{z-\frac{1}{2}l}{r_1^3} - \frac{z+\frac{1}{2}l}{r_2^3}\right]
\end{aligned}
$$

We are actually only interested in the limit for small ***l*** -- i.e. we're interested in the field we obtain in the limit as we hold the product ***lb*** constant while letting ***l*** shrink (and ***b*** increase). To that end, we start by pulling out $\frac{1}{r_1^3}$ from (3) and finding its value for small ***l***:

$$
\begin{aligned}
\frac{1}{r_1^3} &= \frac{1}{\left(y^2+(z-\frac{l}{2})^2\right)^{\frac{3}{2}}} \\
&\underset{l\to 0}{=} \left[\frac{1}{\sqrt{y^2+z^2-zl}}\right]^3 \\
&\underset{l\to 0}{=} \frac{1}{r^3}\left[\frac{1}{\sqrt{1-zl/r^2}}\right]^3 \\
&\underset{l\to 0}{=} \frac{1}{r^3}\left[1+\frac{zl}{2r^2}\right]^3 \\
&\underset{l\to 0}{=} \frac{1}{r^3}\left(1+\frac{3}{2}\frac{zl}{r^2}\right)
\end{aligned}
\tag{4a}
$$

Similarly we have

$$
\frac{1}{r_2^3} \underset{l\to 0}{=} \frac{1}{r^3}\left(1-\frac{3}{2}\frac{zl}{r^2}\right)
\tag{4b}
$$

Plugging (4a) and (4b) back into (3), multiplying out, and discarding all terms in $l^2$, we obtain

$$
\begin{aligned}
\lim_{l\to 0} B_z &= \frac{bl}{r^3}\left[3\frac{z^2}{r^2}-1\right] \\
&= \frac{|\mu|}{r^3}\left(3\cos^2\theta-1\right)
\end{aligned}
\tag{5}
$$

### *The Component $B_y$*

As with $B_z$, we start with the field for a point charge, for which

$$
B_{y[point]} = \left|B_{[point]}\right| \sin\theta
\tag{6}
$$

The net field is the sum of the fields from the two charges, or

$$
\begin{aligned}
B_y &= B_{y(+b)} + B_{y(-b)} \\
&= \frac{b}{r_1^2}\frac{y}{r_1} - \frac{b}{r_2^2}\frac{y}{r_2} \\
&= by\left[\frac{1}{r_1^3}-\frac{1}{r_2^2}\right]
\end{aligned}
\tag{7}
$$

Plugging in (4a) and (4b), we obtain,

(8) $$\lim_{l\to 0} B_y \;=\; \frac{by}{r^3}\left[\frac{3}{2}\frac{zl}{r^2} - \left(-\frac{3}{2}\right)\frac{zl}{r^2}\right]$$

which, with a little rearranging, is

(9) $$\lim_{l\to 0} B_y \;=\; 3\,|\mu|\,\frac{yz}{r^2}\,\frac{1}{r^3}$$

or,

(10) $$\lim_{l\to 0} B_y \;=\; 3\,\frac{|\mu|}{r^3}\,\sin\theta\cos\theta$$

## *The Total Field*

Finally, we put (5) and (10) together to obtain the field at each point. If we use φ for the azimuth -- the angle of rotation about the *z* axis -- then, using the convention that the unit basis vectors are written with hats, we have:

(11) $$\vec{B} \;=\; \frac{3\,|\mu|}{r^3}\left[\sin\theta\cos\theta\,(\cos\phi\,\hat{x} \;+\; \sin\phi\,\hat{y}) \;+\; \left(\cos^2\theta - \tfrac{1}{3}\right)\hat{z}\right]$$

To express the field in polar coordinates, we write the Cartesian basis vectors in terms of the usual orthonormal polar basis:

(12) $$\begin{aligned}\hat{x} &= \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}\\ \hat{y} &= \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}\\ \hat{z} &= \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\end{aligned}$$

The "intermediate bulge" when we multiply out the above expressions will look a little gross but nearly everything will end up canceling. We start by looking just at the term in brackets in (11). We'll take the coefficient on each of the Cartesian basis vectors in turn, multiply it by the corresponding expression in (12), and then, below, we'll sum the terms for each of the polar basis vectors in turn:

Coefficient on $\hat{x}$:

$$\sin\theta\cos\theta\cos\phi[\sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}]$$
$$= \;\cos\theta[\sin^2\theta\cos^2\phi\,\hat{r} + \sin\theta\cos\theta\cos^2\phi\,\hat{\theta} - \sin\theta\sin\phi\cos\phi\,\hat{\phi}]$$

Coefficient on $\hat{y}$:

$$\sin\theta\cos\theta\sin\phi[\sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}]$$
$$= \;\cos\theta[\sin^2\theta\sin^2\phi\,\hat{r} + \sin\theta\cos\theta\sin^2\phi\,\hat{\theta} + \sin\theta\sin\phi\cos\phi\,\hat{\phi}]$$

Coefficient on $\hat{z}$:

$$(\cos^2\theta - \tfrac{1}{3})(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$
$$= \cos\theta\,[\cos^2\theta\,\hat{r} - \sin\theta\cos\theta\,\hat{\theta}] - \cos\theta[\tfrac{1}{3}\,\hat{r} - \tfrac{1}{3}\tan\theta\,\hat{\theta}]$$

Note that the form we chose for the $\hat{z}$ term may look rather odd but it makes the subsequent arithmetic a little easier.

Now we just sum these term by term from the above expressions, with the $\hat{x}$ and $\hat{y}$ expressions summing to form the first term in each of the following expressions and the $\hat{z}$ expression contributing the last two, to obtain:

$\hat{r}$: $$\cos\theta[\sin^2\theta + \cos^2\theta - \tfrac{1}{3}]\,\hat{r}$$
$$= \tfrac{2}{3}\cos\theta\,\hat{r}$$

$\hat{\theta}$: $$\cos\theta[\sin\theta\cos\theta - \sin\theta\cos\theta + \tfrac{1}{3}\tan\theta]\,\hat{\theta}$$
$$= \tfrac{1}{3}\sin\theta\,\hat{\theta}$$

$\hat{\phi}$: $$\cos\theta[0 + 0 + 0]\,\hat{\phi} = 0\,\hat{\phi}$$

Finally we put these together to obtain the polar coordinate expression for the field for a dipole aligned on the Z axis:

(13) $$\vec{B} = \frac{|\mu|}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

We can write the dipole itself as:

(14) $$\vec{\mu} = |\mu|\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$

We scale (14) by $1/r^3$ then add its right hand side and subtract its left hand side from (13) to obtain:

(15) $$\vec{B} = \frac{1}{r^3}\,[3\,|\mu|\,\cos\theta\,\hat{r} - \vec{\mu}]$$

Finally, we can rewrite that in coordinate-free form as:

(16) $$\vec{B} = \frac{1}{r^3}\,[3\,(\vec{\mu}\cdot\hat{r})\,\hat{r} - \vec{\mu}]$$