IMIT
Swedish original by Leif Ek 1974
English verison by Anna Thaning 2002

Optics experiment

Fourier Optics

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1 Introduction

In this experiment we will, as implied by the title, use optical systems to study Fourier transforms. Fourier analysis describes how a function may be divided into a sum (or an integral) of sines and cosines. This is often done in one dimension only, but the optical system provides us with them means to easily do it in two dimensions. It is described how two-dimensional functions (or images), expressed in the spatial coordinates (x, y), are transformed. The two-dimensional Fourier transform is defined as

$$\mathcal{F}\{f(x,y)\} = \iint_{-\infty}^{+\infty} f(x,y)\exp[-\mathrm{i}(k_x x + k_y y)]\mathrm{d}x\mathrm{d}y. \tag{1}$$

Today, two-dimensional Fourier transforms can quite easily be found numerically, for example using image processing software (you can also find a function for it, fft2, in Matlab). But this still requires some time and computational power. During this experiment we will create an instant Fourier transform of a pattern. A diffractive slit in uniform illumination gives rise to a diffraction pattern which turns out to be the fourier transform of the diffractive slit. It is placed at infinity (i.e., very far from the slit) but a focusing lens can be used to move it to the focal plane of the lens. Similarly, a lens may be used to re-transform the image. Schematically, the

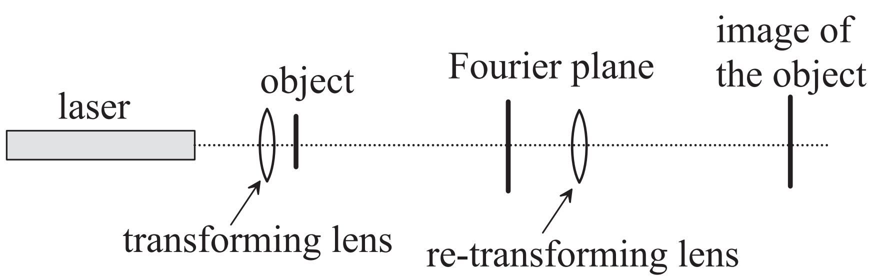


Figure 1: Schematic illustration of the experiment.

setup is shown in Fig. 1. It provides us with the opportunity to view the Fourier transform, but it also gives us the possibility to meddle with it. By blocking some parts of the Fourier transform, we remove certain frequencies from the image. The effect can then be studied in the image of the object. This process is known as *filtering*. Unfortunately, this method for viewing the Fourier transforms has some limitations. The lenses used are not perfect - they have aberrations which will distort the transform. But even with a perfect lens, the true transform will not be found, since the finite size of the lens will cut off the highest frequences. The lens will act as a low-pass filter. The extent of this filtering depends on whether the coherent or the incoherent illumination is used.

The aim of this experiment is to show

1. that Fraunhofer diffraction gives the Fourier transform of the diffracting aperture.
2. what two-dimensional transforms of some simple images look like.
3. the differences that appear for coherent and incoherent illumination.

As an extra bonus, you also get the Babinet's principle...

2 Theory

2.1 Fourier transformation by Fraunhofer diffraction

Now one would think that the Fourier transform on an object, e.g. a square, could be obtained just by illuminating the aperture and then study the diffraction pattern. In reality, it's a little bit more complicated since the diffraction pattern of an aperture changes its shape depending on the distance of observation. Generally, there are two cases: Fresnel diffraction, where the diffraction pattern close to the aperture is considered, and Fraunhofer diffraction, where the pattern is studied at infinity. Fresnel diffraction is normally much more complicated, so we will stick to Fraunhofer diffraction and show that the Fraunhofer diffraction pattern is really the Fourier transform of the object. Assume that the object consists of an aperture, of any

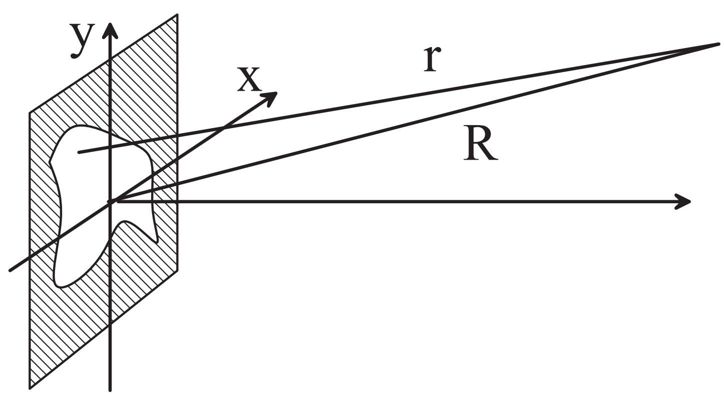


Figure 2: Diffraction from an arbitrary aperture, at $P(X, Y, Z)$.

shape, in the (x, y) plane. We study the diffraction pattern in the point P, of coordinates (X, Y, Z), as illustrated in Fig. 2. Accordning to the Huygens-Fresnel principle, the aperture may be regarded as the sum of many point sources that produce spherical waves. From the point source located in the small surface element $\mathrm{d}S$ we obtain a spherical wave, which gives rise to the electric field $\mathrm{d}E$:

$$\mathrm{d}E = \frac{\epsilon_A}{r} \exp[\mathrm{i}(kr - \omega t)]\mathrm{d}S \tag{2}$$

where ϵ_A is the strength of the source (i.e., the incident field strength at this particular point) and r is the distance between this point on the aperture and the observation point P, as illustrated in Fig. 2. (This is the normal expression for a spherical wave on complex form - see e.g. Eq. 2.75 in Hecht.) To obtain the total field at P, we must sum all the different contributions. This is done by integration:

$$E(P) = \iint_{\mathrm{A}} \frac{\epsilon_A}{r} \exp[\mathrm{i}(kr - \omega t)]\mathrm{d}S, \tag{3}$$

where A is the area of the diffracting aperture. This is a fairly complicated expression, and we need to do some approximations. According to the geometry (see Fig. 2)

$$r = \sqrt{(X-x)^2 + (Y-y)^2 + Z^2} \tag{4}$$

$$R = \sqrt{X^2 + Y^2 + Z^2}. \tag{5}$$

Since the size of the object is small compared to the distance R, we can approximate $r \approx R$ in the denominator. However, due to the large value of kr, very small changes to r will cause large changes to the exponential term, so here we need a more accurate approximation. By some manipulation, we get

$$r = R\sqrt{1 + \frac{x^2+y^2}{R^2} - \frac{xX+yY}{R^2}}. \tag{6}$$

The term $(x^2+y^2)/R^2$ is small and may be neglected. Then expansion in a power series yields

$$r \approx R\sqrt{1 - \frac{2(xX+yY)}{R^2}} \approx R\left[1 - \frac{xX+yY}{R^2}\right]. \tag{7}$$

Insertion of Eq. (7) into Eq. (3) yields

$$E(X,Y,Z) = \iint_{\mathrm{A}} \frac{\epsilon_A}{r} \exp[\mathrm{i}(kr-\omega t)]\mathrm{d}S \approx \iint_{\mathrm{A}} \frac{\epsilon_A}{R} \exp[\mathrm{i}(kr-\omega t)] \exp[-\mathrm{i}k\frac{xX+yY}{R}]\mathrm{d}S. \tag{8}$$

To simplify the expression, we introduce the aperture function

$$A(x,y)) = \frac{\epsilon_A}{R} \exp[\mathrm{i}(kr - \omega t)] = A_0(x,y) \exp[\mathrm{i}\phi(x,y)] \tag{9}$$

which describes the transmittance of the object. It contains an amplitude part $A_0(x,y)$ and a phase part $\phi(x,y)$. R is constant with respect to the integration variables x and y. Eq. (8) may now be rewritten as

$$E(X,Y,Z) = \iint_{-\infty}^{+\infty} A(x,y) \exp[-\mathrm{i}k(xX+yY)/R]\mathrm{d}x\mathrm{d}y. \tag{10}$$

The integration can be extended to infinity, since the aperture function is zero everywhere but in the aperture. In order to identify Eq. (10) as a two-dimensional Fourier transform, we define the spatial frequencies

$$k_x = \frac{kX}{R} \ , \qquad k_y = \frac{kY}{R} \ . \tag{11}$$

Inserted into Eq. (10), they yield

$$E(k_x, k_y) = \iint_{-\infty}^{+\infty} A(x, y) \exp[-\mathrm{i}(k_x x + k_y y)] \mathrm{d}x \mathrm{d}y. \tag{12}$$

Except for the names of the variables, this is identical to Eq. (1). Consequently, we have shown that the electric field at Fraunhofer diffraction is the Fourier transform of the aperture function, i.e., that

$$E(k_x, k_y) = \mathcal{F}\{A(x, y)\}. \tag{13}$$

Since the aperture function describes the geometry of the aperture, this means the diffraction pattern far away from the object will be its Fourier transform.

2.2 Two-dimensional images and transforms

In order to better understand two-dimensional images and their transforms, we'll start with the easier one-dimensional case. Assume we have a function of one variable, $f(t)$, and its transform $F(\omega)$. The function can always be written as a sum of sine and cosine functions, that is, as a sum of different frequency components. The Fourier transform of the function will the be the sum of the Fourier transforms of the trigonometric functions that constitute f(t). In two dimensions, the same thing happens. An image can be viewed as a lot of sinusoidal lattices. These lattices have different spatial frequencies, and different orientations, as illustrated in Fig. 3. The

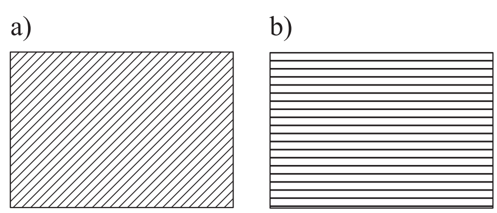


Figure 3: Two sinusoidal gratings of different orientations.

Fourier transform of a sinusoidal lattice is three delta functions, one in the origin and one to each side, as illustrated in Fig. 4. The positions of the the delta pulses are determined by the spatial frequency of the lattice (the higher the frequency, the longer the distance to the origin) and by its orientation. This can easily be deduced mathematically, but we leave this to the interested student.

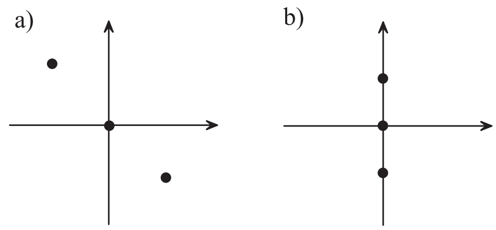
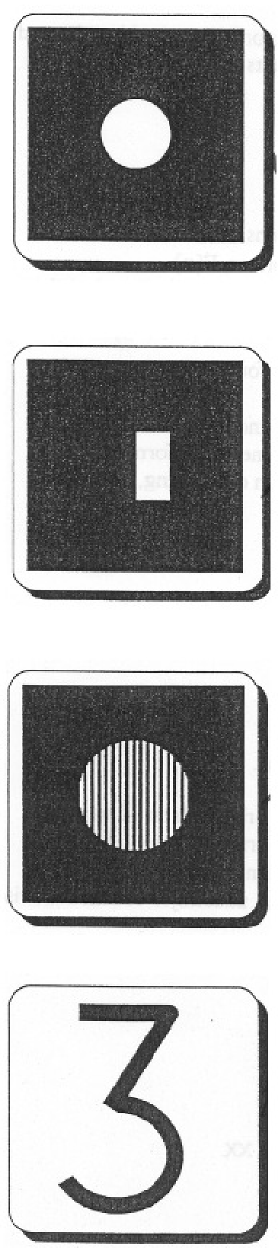
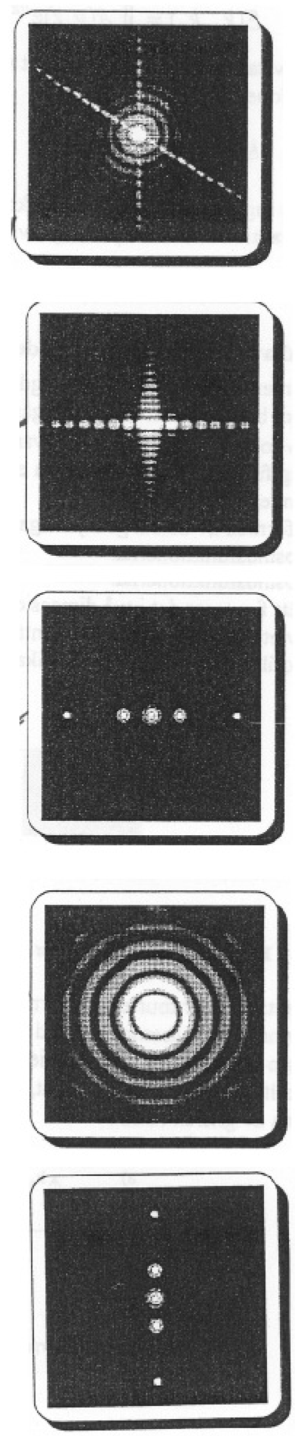


Figure 4: The Fourier transforms of the gratings in Fig. 3.

2.3 Task 1: Some simple transforms

Below, you find four objects (on the left) and five Fourier transforms (on the right). Find one Fourier transform for each object, and mark the connection with a line!





2.4 Transforms of several identical parts

If the object consists of several identical parts, the superposition principle says the image will be the sum of the electrical field from each part. Assume there are N parts, and that each of them gets the local coordinate system (x', y'), where the origin is at (x_j, y_j), $j = 1, ..., N$, as illustrated in Fig. 5. Introduce the notation

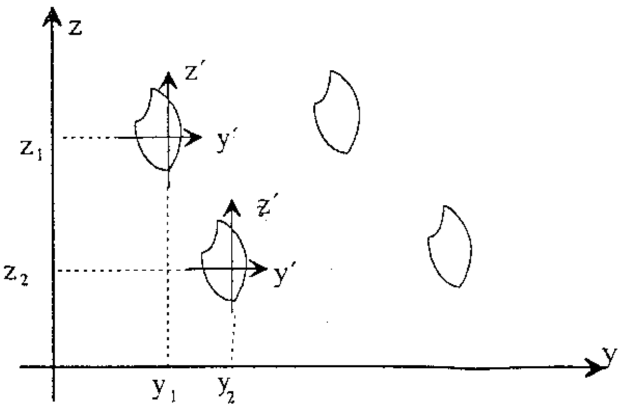


Figure 5: An object consisting of four identical parts.

$A(x, y)$ and $A_1(x', y')$ for the total aperture function and the aperture function for one part, respectively. Then the total field will be

$$E(X,Y) = \sum_{j=1}^{N} \iint_{-\infty}^{+\infty} A_1(x', y') \exp\left[-ik\frac{X(x_j + x') + Y(y_j + y')}{R}\right] \mathrm{d}x'\mathrm{d}y', \quad (14)$$

or

$$E(X,Y) = \iint_{-\infty}^{+\infty} A_1(x', y') \exp\left(ik\frac{Xx' + Yy'}{R}\right) \mathrm{d}x'\mathrm{d}y' \sum_{j=1}^{N} \exp\left(ik\frac{Xx_j + Yy_j}{R}\right). \quad (15)$$

For simplicity, introduce the spatial frequences k_x and k_y as defined in Eq. (11). Then the field my be written as

$$E(k_x, k_y) = \iint_{-\infty}^{'\infty} A_1(x', y') \exp[i(k_x x' + k_y y')]\mathrm{d}x'\mathrm{d}y' \sum_{j=1}^{N} \exp(ik_x x_j) \exp(ik_y y_j). \quad (16)$$

The sum at the end of the expression is a phase factor only - it will not affect the pattern itself. Consequently, several identical parts give the same Fourier transform as one of the parts, except for the extra phase factor.

This shows that the electrical field E at the image plane is the Fourier transform of the aperture function for one part of the pattern A_1, multiplied by the Fourier transform of delta pulses at the origin of each part, i.e., that

$$E = \mathcal{F}\{A(x,y)\} = \mathcal{F}\{A_1(x',y')\} \cdot \mathcal{F}\left\{\sum_{j=1}^{N} \delta(x - x_j)\delta(y - y_j)\right\}. \qquad (17)$$

(Compare to the convolution theorem which says that

$$\mathcal{F}\{A\} = \mathcal{F}\{A_1\} \cdot \mathcal{F}\{g\} = \mathcal{F}\{A_1 * g\}\ .) \qquad (18)$$

Tee conclusion is that several identical parts of a figure may be described as the aperture function of the part, convoluted with delta functions at the centre of each part, as illustrated in Fig 6.

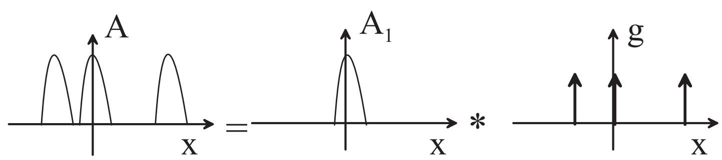


Figure 6: Several identical parts can mathematically be described as a convolution between one part and several delta functions.

2.5 Task 2: The transform of two circles

a) Use two delta functions and a circ function to describe two circles at a distance a from each other. Place one circle at the origin, and the other at $(a, 0)$. A circ function is defined as

$$\text{circ}(x,y) = \begin{cases} 1 & : \quad (x^2+y^2)^{1/2} \le D/2 \\ 0 & : \quad (x^2+y^2)^{1/2} > D/2 \end{cases} \qquad (19)$$

where D is the diameter of the circle.

b) Find the Fourier transform of the two circles! You don't need to calculate the Fourier transform of a single circle (it is an Airy disc, as you probably know) since the phase factor is the interesting part. Then look at Fig. 7. Is this the diffraction pattern of two circles?

c) Find the distance between the two circles, expressed in their diameter D. In the diffraction pattern of a circle, we know that the distance from the centre to the first dark ring is

$$\frac{k_p D}{2\pi} = 1.22\ , \qquad \text{where } k_p = \sqrt{k_x^2 + k_y^2}. \qquad (20)$$

Assume that there are n dark lines inside the circle.

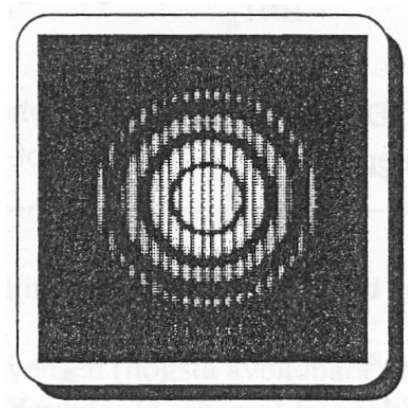


Figure 7: Is this the diffraction pattern of two circles?

2.6 Babinet's principle

Assume there are two complementary objects Σ_1 and Σ_2. Complementary means the first object is dark where the other is transparent, and transparent where the other is dark, as illustrated in Fig. 8. According to Babinet's principle, the diffraction patterns of those objects will be identical. Assume that E_1 is the field behind Σ_1,

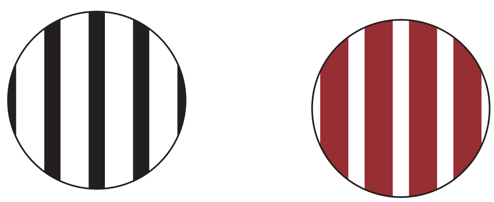


Figure 8: Two complementary objects.

and E_2 the field behind Σ_2. If both objects are illuminated at the same time, no light at all will get through. This means

$$E_1 + E_2 = 0 \quad \Leftrightarrow \quad E_1 = -E_2 \Rightarrow \quad I_1 = I_2 \tag{21}$$

This means the intensities behind the object are identical.

(Because of the finite size of the objects, they are not truly complementary. Because of this, Babinet's principle doesn't apply to the 0:th diffraction order (the centre of the transforms).)

2.7 Transfer theory

In optical imaging there are two main types of illumination, namely *coherent* or *incoherent*. The coherent system is linear with respect to amplitude, and the incoherent is linear with respect to intensity (i.e., for coherent light we add amplitudes, but for incoherent light we add intensities). The main point of this section is to show that the imaging properties of the two systems are different, a fact that will also be confirmed experimentally. Assume there is a point source (x, y) in the ob-

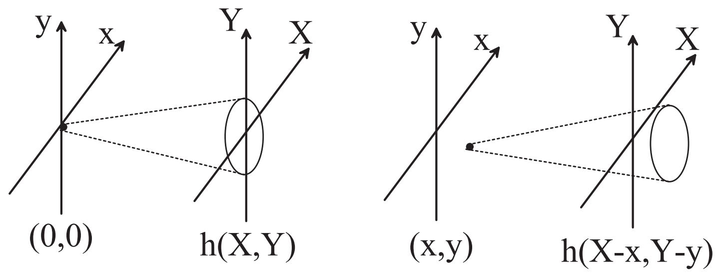


Figure 9: A point source and its distribution in the image plane. When the source is moved, the image is moved in the same way.

ject plane, and that it produces the field $h(X-x, Y-y)$ in the image plane. The corresponding intensity is $|h(X-x, Y-y)|^2$, as illustrated in Fig. 9. If the incident field and intensity is $E(x, y)$ and $I(x, y)$ respectively, the resulting field or intensity in the image plane can be found by adding all those little point sources together (integration). We find that

$$E'(X,Y) = \iint h(X-x, Y-y)E(x,y)\mathrm{d}x\mathrm{d}y \tag{22}$$

for coherent light, and that

$$I'(X,Y) = \iint |h(X-x, Y-y)|^2 I(x,y)\mathrm{d}x\mathrm{d}y \tag{23}$$

for incoherent light. Since both expressions are convolutions, they can be Fourier transformed to yield

$$\mathcal{F}\{E'\} = \mathcal{H} \cdot \mathcal{F}\{E\} \tag{24}$$

$$\mathcal{F}\{I'\} = \mathcal{H} * \mathcal{H} \cdot \mathcal{F}\{I\} \tag{25}$$

where $\mathcal{H} = \mathcal{F}\{h\}$. $\mathcal{H}$ and $\mathcal{H}*\mathcal{H}$ are the *transfer functions* of the system. For an ideal lens, $\mathcal{H}$ is illustrated in Fig. 10. This transfer function (a flattop) is valid for coherent light. For incoherent light two flattops should be convoluted, to give the triangular transfer function of Fig. 11. It's worth noticing that the limiting frequency (the highest frequency that can be imaged by the system) for the incoherent light is twice that of the coherent light. Incoherent light is better for imaging high frequencies, while the lower frequencies are imaged better in coherent light.

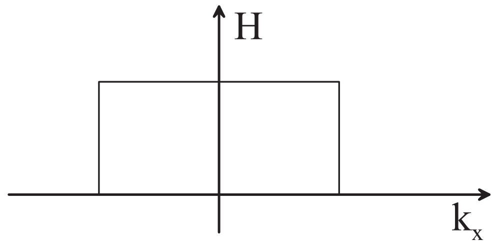


Figure 10: The transfer function for an ideal lens in coherent illumination.

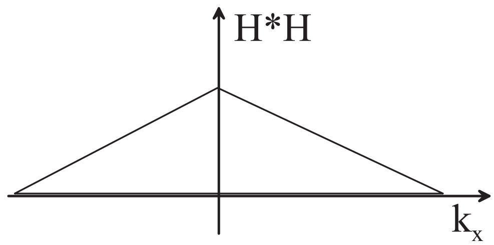


Figure 11: The transfer function for an ideal lens in incoherent illumination.

2.8 Summary of the theory

After reading the instructions, and before performing the experimental tasks, you should know the following:

1. That the Fourier transform of an object may be found through Fraunhofer diffraction

2. That an object containing several identical patterns will give the Fourier transform of this pattern, modulated by an interference pattern.

3. Accordning to Babinet's principle, two complementary objects give the same intensity pattern.

4. That transfer functions can be used to see what spatial frequencies a system will image.

5. That coherent and incoherent systems have different transfer functions, and consequently different imaging properties.

3 Experiment

3.1 Equipment

The basic equipment for this experiment is displayed in Fig. 12. A HeNe laser

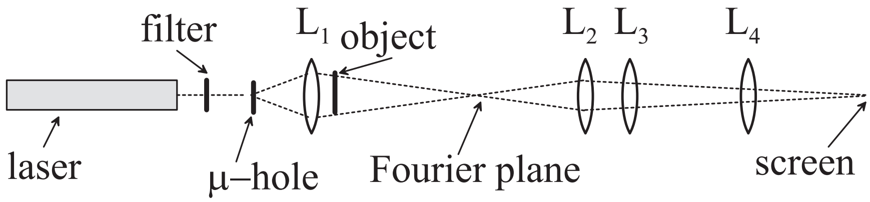


Figure 12: Schematic illustration of the experiment. L_1 and L_2 give a magnified image of the Fourier plane. L_4 re-transforms it into an image of the object, which can be seen on the screen.

provides coherent illumination, via a microscope objective (with a micrometer hole) that creates a spherical wave. Then lens L_1 performs the Fourier transformation, and a holder is placed at the Fourier plane. Two lenses L_2 and L_3 provide a magnification of the Fourier transform on the screen. The lens L_4 will be inserted and removed many times during the experiment, since it projects the image on the screen. For incoherent illumination, the laser is replaced by a lamp.

Never look into the laser beam!

The intensity is high enough to cause eye damage.

Don't touch the lenses or the objects, other than at the edges.

3.2 Tasks

Setting up the experiment

part 1

- Turn the laser on.
- Move the μ-hole in the transverse direction until the transmitted intensity is maximized.
- Move the μ-hole towards the microscope objective. Be careful that you don't move them close enough to touch each other.
- Move the μ-hole in the transverse direction again. Repeat the procedure until the hole is in the focal point of the microscope objective (i.e., when there is no interference pattern in the transmitted intensity).
- Insert lens L_1.
- Place a diffusely reflecting surface (a piece of aluminum) in the x-y-holder and find the Fourier plane by maximizing the speckle size.

* Insert lenses L_2 and L_3 so that the Fourier plane is magnified on the screen. Use for example object 9.
* Insert lens L_4 so that the object is imaged on the screen.

Some simple transforms

part 2

* Place object 1 in the object holder. It consists of four geometrical patterns. Look at their images on the screen!
* Block all patterns except one at the object. Then view the Fourier transform on the screen (i.e., remove L_4). The transform is difficult to see, so you need almost complete darkness. There might be a better version of the Fourier pattern at the x-y-holder.
  Task: Observe and draw the transforms of all four patterns.

Several identical images

part 3

* Insert object 2 (two circular holes). Observe the Fourier transform.
  Task: Draw the Fourier transform of two circles. Does it agree with your predictions in task 2 of the theory section? use the Fourier transform to determine the distance between the circles, expressed in their diameter. Check your results by measuring the distance in the image of the object! Think of possible causes for errors.

part 4

* Replace object 2 by object 3 (three circles). Try to predict the Fourier transform, then look at it. Was your prediction accurate?
  Task: Draw the Fourier transform.

part 5

* Insert object 4 (random circles).
  Task: Draw the Fourier transform. Explain what you see!

part 6

* Insert object 5 (random circles).
  Task: Draw the Fourier transform. Explain what you see. Are the circles really random?

Babinet's principle

part 7

* Insert object 6 (two gratings).

* Insert an aluminum piece that allows you to block the left or the right side of the object.
  Task: Regard the two Fourier transforms (from the left and the right part). Are they different from each other?
  Task: Remove the aluminum piece, and instead block the 0:th order in the Fourier plane (make a black dot on a piece of glass). Look at the image of the object. What has happened? Explain!
  Task: Now block everything except the 0:th order in the Fourier plane (make a small hole in a piece of paper). Look at the image, and explain what has happened.
  Task: Make a slightly bigger hole, which lets orders 0 and +1 through. Explain what you see.

Frequency filtering

part 8
* Insert object 7 (a sector star).
* Image the star on the screen. Then watch the Fourier transform. Different parts of the Fourier transform correspond to different parts of the image. Try to understand how they are connected!
  Task: Around order 0 in the Fourier tranform, there is a dark area. Why?
  Task: Perform low-pass and high-pass filtering, using holes and dots. Regard the image of the object, and explain what happens!
  Task: Use a piece of paper to block the lower half of the Fourier plane, but place it low so that the central part of the transform can pass through. The use the x-y-holder to move the paper slowly, until the central part is blocked too. Watch the image of the object while doing this. Explain what happens!

Transfer theory

part 9
* Use the same object (7), but perform low-pass filtering in the Fourier plane, so that the limiting frequency is at half the radius of the star.
* Insert filters in front of the object, to reduce the intensity of the light.
* ONLY CONTINUE IF THE ASSISTANT IS PRESENT!
* The assistant will check that the intensity level is low enough.

* Look into the system through the last lens L_4. You will see an image of the sector star. (You can replace L_4 by a telescope if you want to.)
* NOBODY SHOULD TOUCH THE SYSTEM WHILE SOMEBODY IS LOOKING INTO IT! Changes might lead to sudden changes in intensity level.
* Insert the diffusing glass behind the object. This is allowed, since it will only reduce the intensity.
* Increase the intensity slowly, by moving the filter, until the person watching can see a speckle pattern.
* Move the diffusing glass back and forth in the holder (careful not to un-block the opening). Now you have made the light incoherent.

  Task: What has happened to the image and the limiting frequency? Which was better for imaging the high frequences, the coherent or the incoherent light?

part 10

* Replace object 7 by object 8, remove all filters and the diffusing glass and insert L_4. Object 8 consists of two gratings, whose amplitude transmittance is shown in Fig. 13.

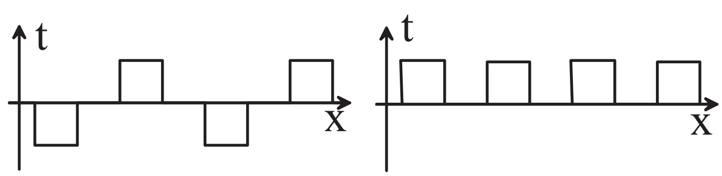


Figure 13: The amplitude transmittance of the two gratings in object 8.

  Task: What is the intensity transmittance of the two gratings?
* Observe each grating and their Fourier transforms, both separately (block part of the image) and together.

  Task: Explain the differences between the two Fourier transforms! Identify the two gratings. This is a difficult task - ask your assistant if you can't solve it.
* Remove the aluminum piece that blocks half the image, and insert a filter in the Fourier plane. It should, just barely, let three orders pass.

  Task: Then move the filter in the Fourier plane, along the Fourier transform, and see what happens to the image. Explain!

Filters and pattern recognition

part 11

* Insert object 9 (different gratings).
* Watch the image and the Fourier transform. Try to explain how different parts of the Fourier transform correspond to different parts of the image.
* Insert a filter that lets only order 0 through.
  Task: Move the hole using the x-y-holder, so that different parts of the Fourier transform are transmitted. Watch the image! Can you explain what happens to it?
  Task: Make a filter in the Fourier plane, which removes all horizontal gratings but not the others.

part 12

* Insert object 12 (three circles).
* Observe the image of the object. What do you think the Fourier transform will look like? Look at the Fourier transform to see if you were right!
  Task: Explain the Fourier transform!
  Task: Make a filter on the Fourier plane, which removes two of the circles but lets the third one through!

part 13

* Keep object 12 but remove the filter.
* Turn the laser off, and introduce white light. Don't change the system, just introduce the white light through a mirror at $\sim 45°$. This way, you can remove the mirror and turn the laser on, and the system will still be aligned.
* Regard the image using L_4. You might need to insert some filters.
  Task: Make the three circles appear in different colours, for example one red, one yellow and one blue!

part 14

* If there is still time, there are extra tasks for those interested. Ask the assistant!