20

The Skin Effect

20.1 Introduction

We are now in a position to formulate any electromagnetic problem in terms of Maxwell's equations. This chapter deals with the skin effect, the first practical electromagnetic problem we will solve as an example of this kind. A related effect, the proximity effect, is then considered briefly.

We know that a time-invariant current in a homogeneous cylindrical conductor is distributed uniformly over the conductor cross section. If the conductor is not cylindrical, the time-invariant current in it is not distributed uniformly, but it exists in the *entire* conductor. We shall see in this chapter that a time-varying current has a tendency to concentrate near the surfaces of conductors. If the frequency is very high, the current is restricted to a very thin layer near the conductor surfaces, practically on the surfaces themselves. Because of this extreme case, the entire phenomenon of nonuniform distribution of time-varying currents in conductors is known as the *skin effect*.

The cause of the skin effect is electromagnetic induction. A time-varying magnetic field is accompanied by a time-varying induced electric field, which in turn creates secondary time-varying currents (induced currents) and a secondary magnetic field. We know from Lentz's law that the induced currents produce the magnetic flux, which is opposite to the external flux (which "produced" the induced currents), so that the total flux is reduced. The larger the conductivity, the larger the induced currents are, and the larger the permeability, the more pronounced is the flux reduction.

Consequently, both the total time-varying magnetic field and induced currents inside conductors are reduced when compared with the dc case.

The skin effect is of considerable practical importance. For example, at very high frequencies a very thin layer of conductor carries most of the current, so we can coat any conductor with silver (the best available conductor) and have practically the entire current flow through this thin silver coating. (Unfortunately silver oxidizes easily, so gold is often used instead because it is inert.) Even at low, power-line frequencies (60 Hz in the United States and Canada, and 50 Hz in Europe), in the case of high currents the use of thick, solid conductors is not efficient; bundled conductors are used instead.

The skin effect exists in all conductors, but as mentioned, the tendency of current and magnetic flux to be restricted to a thin layer on the conductor surface is much more pronounced for a ferromagnetic conductor than for a nonferromagnetic conductor of the same conductivity. For example, for iron at 60 Hz the thickness of this layer is on the order of only 0.5 mm. Consequently, solid ferromagnetic cores for alternating current electric motors, generators, transformers, etc., would have very high eddy-current losses. Therefore laminated cores made of thin, mutually insulated sheets are used instead. At very high frequencies, ferrites (ferrimagnetic materials) are used because they have very low conductivity when compared to metallic ferromagnetic materials.

Questions and problems: Q20.1 to Q20.10

20.2 Skin Effect

Consider an idealized case of a sinusoidal current in a homogeneous conducting half-space, as sketched in Fig. 20.1. Let the angular frequency of the current be ω and let the medium have a conductivity σ and permeability μ. Finally, assume that the

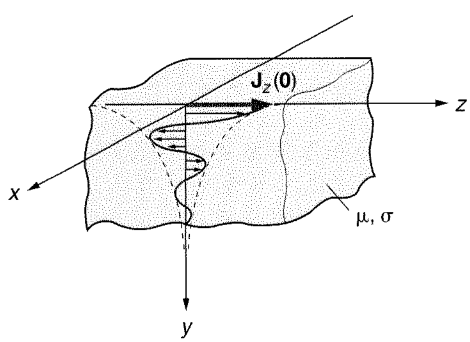

Figure 20.1 Homogeneous conducting half-space with sinusoidal current. The amplitude of the current density vector at an instant of time versus the distance y from the boundary surface is as indicated.

current density vector is parallel to the boundary surface, and that it has a single component, for example, $\mathbf{J} = J_z \mathbf{u}_z$, depending on the coordinate y (the distance from the interface) only. We wish to determine the distribution of current in the conducting half-space.

At first glance, one might be tempted to think this problem is purely academic. We will see, however, that it has important practical implications. After solving Maxwell's equations, we will find that the intensity of the current density vector and of all the field vectors decreases exponentially with the distance from the boundary surface. This decrease is more rapid at higher frequencies and for higher conductivities and permeabilities. For conductors used in everyday practice (copper, for instance), and frequencies higher than about 1 MHz, the thickness of the current layer becomes less than a fraction of a millimeter. If we consider *any* conductor whose radius of curvature is much larger than the current layer thickness, the results we will obtain can be applied with high accuracy. Therefore this section has considerable practical importance and applicability.

We start the analysis from the differential form of Maxwell's equations in complex form. Because we assume the medium to be a good conductor, the displacement current density in the second equation can be neglected. We start from

$$\nabla \times \mathbf{E} = -\mathrm{j}\omega \mathbf{B} \qquad \nabla \times \mathbf{H} = \mathbf{J}. \tag{20.1}$$

For simplicity we do not underline the complex vectors $\mathbf{E}$, $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{J}$. Since $\mathbf{E} = \mathbf{J}/\sigma$ and $\mathbf{H} = \mathbf{B}/\mu$, Eqs. (20.1) become

$$\nabla \times \mathbf{J} = -\mathrm{j}\omega\sigma \mathbf{B} \qquad \nabla \times \mathbf{B} = \mu \mathbf{J}. \tag{20.2}$$

We assumed that the current density vector has only a z component, which depends only on y. From the Biot-Savart law and symmetry it therefore follows that there is only an x component of the vector $\mathbf{B}$. According to the expression for the curl in a rectangular coordinate system, Eqs. (20.2) become

$$\frac{\mathrm{d}J_z}{\mathrm{d}y} = -\mathrm{j}\omega\sigma B_x \qquad -\frac{\mathrm{d}B_x}{\mathrm{d}y} = \mu J_z. \tag{20.3}$$

We use ordinary derivatives (not partial derivatives) because J_z and B_x depend only on y.

From Eqs. (20.3) we can eliminate B_x to obtain an equation in J_z:

$$\frac{\mathrm{d}^2 J_z}{\mathrm{d}y^2} = \mathrm{j}\omega\mu\sigma J_z. \tag{20.4}$$

This equation has a simple solution,

$$J_z(y) = J_1 \mathrm{e}^{Ky} + J_2 \mathrm{e}^{-Ky}, \tag{20.5}$$

where

$$K = \sqrt{\mathrm{j}\omega\mu\sigma} = (1+\mathrm{j})\sqrt{\frac{\omega\mu\sigma}{2}} = (1+\mathrm{j})k \qquad k = \sqrt{\frac{\omega\mu\sigma}{2}}. \tag{20.6}$$

Assume that for $y = 0$ the current density is $J_z(0)$. For $y \to \infty$, the current density cannot increase indefinitely, so $J_1 = 0$. Thus we finally have

$$J_z(y) = J_z(0)e^{-ky}e^{-jky}. \tag{20.7}$$

The intensity of the current density vector decreases exponentially with increasing y. At a distance

$$\delta = \frac{1}{k} = \sqrt{\frac{2}{\omega\mu\sigma}} \quad \text{(m)}, \tag{20.8}$$

(Definition of skin depth)

the amplitude of the current density vector decreases to $1/e$ of its value $J_z(0)$ at the boundary surface. This distance is known as the *skin depth*.

As mentioned, although derived for a special case of currents in a half-space, the preceding analysis is valid for a current distribution in any conductor whose radius of curvature is much larger than the skin depth.

Example 20.1—Skin depth for some common materials. As an illustration, let us determine the skin depth for copper ($\sigma = 57 \cdot 10^6\,\text{S/m},\ \mu = \mu_0$), iron ($\sigma = 10^7\,\text{S/m},\ \mu_r = 1000$), seawater ($\sigma = 4\,\text{S/m},\ \mu = \mu_0$), and wet soil ($\sigma = 0.01\,\text{S/m},\ \mu = \mu_0$) at 60 Hz (power-line frequency), 10^3 Hz, 10^6 Hz, and 10^9 Hz. The results are summarized in Table 20.1. Note that for iron the skin depth is very small (significantly less than a millimeter) even at the low power-line frequency. For seawater, the power-frequency skin depth is also relatively small (about 35 m), and for a radio frequency of 1 MHz it is less than 25 cm. For copper, at 1 MHz the skin depth is less than one-tenth of a millimeter.

Example 20.2—Why not use cheap iron instead of expensive copper for distributing electric power? The skin depth for iron at 60 Hz in Table 20.1 answers an important question. If iron has a conductivity that is only about six times less than that of copper, and copper is much more expensive than iron, why do we not use iron wires to carry electric power to our homes? With the millions of kilometers of such wires, that would mean very large savings.

Unfortunately, due to its large relative permeability, iron has a very small skin depth at powerline frequency, so the losses in iron wire are large, outweighing the savings. Thus we have to use copper or aluminum.

TABLE 20.1 Skin depth (δ) for some common materials

Material	$f = 60$ Hz	$f = 10^3$ Hz	$f = 10^6$ Hz	$f = 10^9$ Hz
Copper	8.61 mm	2.1 mm	0.067 mm	2.11 μm
Iron	0.65 mm	0.16 mm	5.03 μm	0.016 μm
Seawater	32.5 m	7.96 m	0.25 m	7.96 mm
Wet soil	650 m	159 m	5.03 m	0.16 m

Example 20.3—Mutual inductance between cables laid on the bottom of the sea. Assume we have three single-phase 60-Hz cables laid at the bottom of the sea (for example, to supply electric power to an island). The cables are spaced by a few hundred meters and are parallel. (Three distant single-phase instead of one three-phase cable are often used for safety reasons: if a ship accidentally pulls and breaks one cable with an anchor, two are left. In addition, usually a spare single-phase cable is laid to enable quick replacement of a damaged one.) If the length of the cables is long (in practice, it can be many kilometers), we might expect very large mutual inductance between these cables, due to the huge loops they form. The skin depth of seawater at 60 Hz (Table 20.1), however, tells us that there will be practically *no* mutual inductance between the cables.

According to Eq. (20.7), with increasing y the current density changes not only in amplitude *but also in phase*. Thus, at a distance $y = \pi/k$ from the boundary surface the vector **J** has at all times *the opposite actual direction* to that near the boundary surface. The distribution of current density as a function of y at an instant is sketched in Fig. 20.1.

An important problem that we are now ready to solve is Joule's losses in the conductor per unit area of the boundary surface. Because we know the current density vector, one possibility is to integrate $[|J_z^2(y)|/\sigma]\,\mathrm{d}y$ from $y = 0$ to infinity, which is not too difficult to do. There is an easier way, however, that does not require integration but uses the concept of the Poynting vector. This derivation is given as problem P20.7 at the end of the chapter. Here we quote and discuss the final result of this derivation. It is found that the power of Joule's losses and the internal reactive power inside the conductor, per area S, are given by

$$P_J = \int_S R_s |H_0|^2 \mathrm{d}S = (P_{\text{reactive}})_{\text{internal}} \quad \text{(W)}, \tag{20.9}$$

(Evaluation of Joule's losses and reactive power in conductors at high frequencies)

where H_0 is the complex rms value of the tangential component of the vector **H** *on the conductor surface*. (By assumption, the normal component of **H** does not exist.) R_s is the *surface resistance* of the conductor, given by

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}} \quad (\Omega). \tag{20.10}$$

(Definition of surface resistance)

This formula for the surface resistance can be obtained if we consider the following rough approximation, illustrated on the square metal slab in Fig. 20.2. We assume that the entire high-frequency current is flowing uniformly over the cross

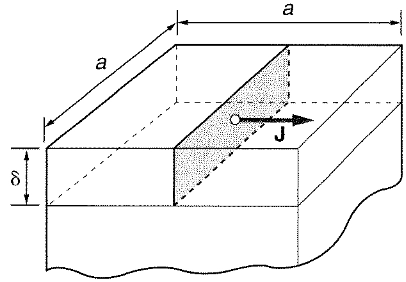

Figure 20.2 The surface resistance of a conductive slab with a uniform current flowing through a cross section δ deep and a wide

section defined by the skin depth and the width a of the conductor slab. Then the resistance of the slab is given by

$$R = \frac{1}{\sigma}\frac{a}{a\delta} = \frac{1}{\sigma}\sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{\omega\mu}{2\sigma}},$$

which can be obtained in an exact manner using the complex Poynting's vector.

Equation (20.9) is used to determine the attenuation in all metallic systems for guiding electromagnetic energy, like two-wire lines, coaxial lines, and rectangular waveguides. We illustrate this with two examples.

Example 20.4—Resistance and internal inductance per unit length of a cylindrical wire at high frequencies. Consider a straight round wire of radius a, conductivity σ, and permeability μ, carrying a sinusoidal current of angular frequency ω and with an rms value I. The magnetic field intensity on the wire surface is $H(0) = I/(2\pi a)$, so the Joule's losses per unit length of the wire, according to Eq. (20.9), are

$$P_J{}' = R_s \frac{I^2}{(2\pi a)^2} 2\pi a = R_s \frac{I^2}{2\pi a}.$$

Because the resistance per unit length is defined by the relation $P_J{}' = R'I^2$, we obtain that, at high frequencies,

$$R' = \frac{R_s}{2\pi a} \quad (\Omega/\text{m}). \tag{20.11}$$

(Resistance per unit length of round conductor at high frequencies)

According to Eq. (20.9), the reactive power at high frequencies inside the conductor per unit area is the same as the power of Joule's losses. We know from circuit theory that the internal reactive power per unit length of the wire can be expressed as $X'_{\text{int}}I^2$. The power in Eq. (20.9) refers to the field *inside the conductor* (i.e., the wire) *only*. Since it is positive, the internal reactance is inductive, that is, $X'_{\text{int}} = R' = \omega L'_{\text{int}}$. Therefore the *internal inductance* of the wire at high frequencies per unit length is given by

$$L'_{\text{int}} = \frac{R'}{\omega} = \frac{R_s}{2\pi a\omega} \quad \text{(H/m)}. \tag{20.12}$$

(Internal inductance per unit length of round conductor at high frequencies)

This formula for L'_{int} was given in Table 18.1.

Example 20.5—Resistance and internal inductance per unit length of a thin two-wire line at high frequencies. Using the results from Example 20.4, it is a simple matter to calculate the resistance and internal inductance per unit length of a thin two-wire line. Let the line have conductors of radius a, and let the distance between the wires be much larger than a. Then the influence of the current in one wire on the current distribution inside the other can be neglected. That means that the current distribution in each wire is practically axially symmetric, as for a single wire in Example 20.4. Therefore, the resistance and internal inductance per unit length are just twice those calculated in the preceding example (because we have two wires). This is the formula given in Table 18.1.

Questions and problems: Q20.11 and Q20.12, P20.1 to P20.12

20.3 Proximity Effect

The term *proximity effect* refers to the influence of alternating current in one conductor on the current distribution in another, nearby conductor. Qualitatively, it can also be explained by Lentz's law.

Consider a coaxial cable of finite length. Assume for the moment that there is an alternating current only in the inner conductor (for example, that it is connected to a generator), and that the outer conductor is not connected to anything. If the outer conductor is much thicker than the skin depth, there is practically no magnetic field inside the outer conductor. If we apply Ampère's law to a coaxial circular contour contained in that conductor, it follows that the induced current on the *inside* surface of the outer conductor is exactly equal and opposite to the current in the inner conductor. This is an example of the proximity effect.

The current from the inner surface of the outer conductor must close into itself over the *outside* surface of the outer conductor, so that on that surface the same current exists as in the inner conductor.

Let us now, in addition, have normal cable current in the outer conductor. It is the same, but opposite, to the current on the conductor outer surface, so the two cancel out. We are left with a current over the inner conductor, and a current over the inside surface of the outer conductor. This is a combined skin effect and proximity effect. Normally, this *combined* effect is what is actually encountered, but it is usually called just the proximity effect.

If the skin effect is not pronounced, the situation is similar except that there is an appreciable current density at all points of the inner and outer cable conductors, as sketched in Fig. 20.3.

The analysis of the proximity effect (i.e., of the combined proximity effect and skin effect) is rather complicated. We shall not, therefore, illustrate the proximity ef-

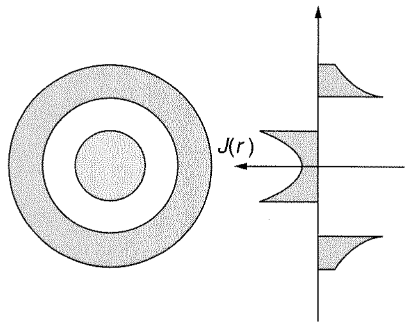

Figure 20.3 Combined proximity and skin effects in a coaxial cable

fect for the general case. If the skin effect is very pronounced, however, in some cases it becomes quite simple, as the next example shows.

Example 20.6—Resistance and internal inductance per unit length of a coaxial cable at high frequencies. Consider again a coaxial cable, with an inner conductor of radius a and an outer conductor of inner radius b. Assume that a sinusoidal current of rms value I flows through the cable at a frequency for which the skin effect is very pronounced. In this case, we have two thin current layers: one over the inner conductor, and one over the inside surface of the outer conductor, as explained previously. According to Eq. (20.9), Joule's losses per unit length of the cable (in both conductors) are given by the sum of losses in the cylinders of radius a and of radius b. Therefore, the resistance per unit length is the sum of that in Eq. (20.11) and of the same expression with a substituted by b:

$$R' = \frac{R_s}{2\pi a} + \frac{R_s}{2\pi b} = \frac{R_s}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right). \tag{20.13}$$

We know that the internal reactance per unit length has the same value as R'. The internal inductance of the cable at high frequencies is therefore $L'_{\text{int}} = R'/\omega$. These are the formulas given earlier in Table 18.1 without explanation.

20.4 Chapter Summary

1. Sinusoidal currents in good conductors are not distributed uniformly over their cross section. Rather, as frequency increases, the current tends to concentrate near the conductor surfaces, a phenomenon known as the *skin effect.*
2. At very high frequencies, the skin effect is so pronounced that current exists only over a very thin layer of any good (metallic) conductor.
3. The penetration of current in a good conductor is characterized by the *skin depth.* At this depth, the current density is $1/e \simeq 0.37$ of that at the conductor surface. At 60 Hz, it is on the order of 1 cm for copper and 1 mm for iron.
4. The skin depth is inversely proportional to the square root of frequency, permeability, and conductivity.

5. A time-varying current in one conductor influences the current distribution in nearby conductors, a phenomenon known as the proximity effect.
6. Both skin effect and proximity effect are consequences of electromagnetic induction.
7. In conducting magnetic materials, the time-varying magnetic field also exhibits the skin effect. For this reason, ferromagnetic cores of alternating-current machinery are made of thin, mutually insulated sheets. At very high frequencies, transformer and inductor cores are made of ferrites, which have a relatively high permeability, but are also relatively good insulators, so that the skin effect for the magnetic field almost does not exist.

QUESTIONS

Q20.1. Three long parallel wires a distance d apart are in one plane. At their ends they are connected together. These common ends are then connected by a large loop to a generator of sinusoidal emf. Are the currents in the three wires the same? Explain. [*Hint:* Have in mind Eq. (14.3), where $\mathbf{J}\,dv$ is substituted by $i\,d\mathbf{l}$.]

Q20.2. N long parallel thin wires are arranged uniformly around a circular cylinder. At their ends the wires are connected by a large loop to a generator of sinusoidal emf. Are the currents in the N wires the same? Explain.

Q20.3. Another wire is added in question Q20.2 along the axis of the cylinder. Is the current in the added wire the same as in the rest? Is it smaller or greater? Explain, having in mind Eq. (14.3).

Q20.4. A thin metallic strip of width d carries a sinusoidal current of a high frequency. What do you expect the distribution of current in the strip to be like?

Q20.5. The two conductors of a coaxial line are connected in parallel to a generator of sinusoidal emf. Is the current intensity in the two conductors the same? If it is not, does the difference depend on frequency? Explain.

Q20.6. A metal coin is situated in a time-harmonic uniform magnetic field, with faces normal to the field lines. What are the lines of eddy currents in the coin like? What are the lines of the induced electric field of these currents?

Q20.7. So-called induction furnaces are used for melting iron by producing large eddy currents in iron pieces. Assume that the iron in the furnace is first in the form of small ferromagnetic objects (nails, screws, etc.). What do you expect to happen if they are exposed to a very strong time-harmonic magnetic field? What happens once they melt?

Q20.8. Two parallel, coplanar thin strips carry equal time-harmonic currents. What do you think the current distribution in the strips is like if the currents in the strips are (1) in the same direction, and (2) in opposite directions?

Q20.9. A thick copper conductor of square cross section carries a large time-harmonic current. Where do you expect the most intense Joule's heating of the conductor? Explain.

Q20.10. A ferromagnetic core of a solenoid is made of thin sheets. If the current in the solenoid is time-harmonic, where do you expect the strongest heating of the core due to eddy currents?

Q20.11. Describe the procedure of determining the resistance and internal inductance per unit length of a stripline at high frequencies. Neglect edge effect.

Q20.12. When compared with current density on the surface, what is the magnitude of current density in a thick conducting sheet one skin depth below the surface, and what is it at two skin depths below the surface?

PROBLEMS

P20.1. Check all skin depth values given in Table 20.1.

P20.2. Starting from Eq. (20.7), prove that the total current in the half-space in Fig. 20.1 is the same as if a current of constant density $J_z(0)/(1+\mathrm{j})$ exists in a slab $0 \le y \le \delta$.

P20.3. Determine the total Joule's losses per unit area of the half-space in Fig. 20.1 by integrating the density of Joule's losses. Compare the result with Eq. (20.9).

P20.4. Using Poynting's theorem in complex form, prove that for any conductor with two close terminals, at very high frequencies the conductor resistance and internal reactance are equal. Find the (integral) expression for these quantities.

P20.5. A stripline of strip width $a = 2\,\text{cm}$, distance between them $d = 2\,\text{mm}$, and the thickness of the strips $b = 1\,\text{mm}$ carries a time-harmonic current of rms value $I = 0.5\,\text{A}$ and frequency $f = 1\,\text{GHz}$. The strips are made of copper. Neglecting fringing effect, determine the line resistance and total inductance per unit length.

P20.6. Starting from Eqs. (20.3), determine the distribution of current in a flat conducting sheet of thickness d. The sheet conductivity is σ, permeability μ, and angular frequency of the current is ω. Set the origin of the y coordinate at the sheet center, and assume that the rms value of the current density at the center is $J_z(0)$. Plot the resulting current distribution.

***P20.7.** Find $H_x(y)$ from Eqs. (20.3) and (20.7), and $E_z(y)$ from Eq. (20.7). Use these expressions and Poynting's theorem to prove Eq. (20.9).

P20.8. Starting from Eq. (20.7), derive the expression for the instantaneous value of the current density, $J_z(y,t)$.

P20.9. Calculate the resistance per unit length of a round copper wire of radius $a = 1\,\text{mm}$, from the frequency for which the skin depth is one-tenth of the wire radius, to the frequency $f = 10\,\text{GHz}$. Plot this resistance as a function of frequency.

P20.10. Assume that in a ferromagnetic round wire of radius a, conductivity σ, and permeability μ, there is an axial magnetic field of angular frequency ω and of rms flux density B practically constant over the wire cross section. Find the expressions for eddy currents in the wire and eddy current losses in the wire per unit length.

P20.11. A bunch of N insulated round wires of radius a, conductivity σ, and permeability μ is exposed to an axial time-harmonic magnetic field of angular frequency ω. The frequency is sufficiently low that the field can be considered uniform over the cross section of the wires. If the rms value of the magnetic flux density is B_0, determine the time-average eddy current power losses in the bunch, per unit volume *of the wires*. Use the result of the preceding problem. Specifically, calculate the losses per unit volume assuming $B_0 = 0.1\,\text{T}$, $a = 0.5\,\text{mm}$, $\sigma = 10^7\,\text{S/m}$, $\mu = 1000\mu_0$, and $f = 60\,\text{Hz}$.

***P20.12.** Consider a straight wire of radius a, conductivity σ, and permeability μ. Let the wire axis be the z axis of a cylindrical coordinate system. Assume there is a current in the wire of rms value I and angular frequency ω. Starting from Maxwell's equations in cylindrical coordinates, derive the differential equation for the only existing, J_z component of the current density vector in the wire. Note that, by symmetry, the only component of **H** is H_ϕ. Do *not* attempt to solve the equation you obtain. (If your equation is correct, it is known as a Bessel differential equation, and its solutions are known as Bessel functions.)